

The calculation of multi-fractal properties of directed random walks on hierarchic trees with continuous branching

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We consider the hierarchic tree Random Energy Model with continuous branching and calculate the moments of the corresponding partition function. We establish the multifractal properties of those moments. We derive formulas for the normal distribution of random variables, as well as for the general case. We compare our results for the moments of partition function with corresponding results of logarithmic 1-d REM and conjecture a specific powerlaw tail for the partition function distribution in the high-temperature phase. Our results establish a connection between reaction-diffusion equations and multi-scaling.

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I. INTRODUCTION

Random energy models on hierarchic trees are rather well-known and much investigated objects of statistical physics [1],[2],[3],[4]. The tree is constructed via a deterministic branching process, starting from the root node and adding q branches from every node of the tree, so that after K steps of the procedure there are q^K nodes of the last generation (the endpoints of the tree). One then associates random energy variables to every branch of the tree, and further attributes the random energy to every node of the last generation by adding up all the energy variables along the unique path connecting the given endpoint to the root. At the last step of the procedure one builds up the partition function [1]-[3]. In the context of statistical physics the model has been identified with directed polymers on the trees, and has been also related to the spin-glass model REM [5],[6]. Independently, similar models have been applied and extensively investigated also in the context of financial mathematics and turbulence [7],[8],[33],[10],[11]. They are also intimately connected to 2-d conformal models [12],[13]. The hierarchical structure of the tree naturally induces recursive relations for the partition function, with the branching number q being a parameter in the recursion. While by construction the branching number q is an integer, one can formally consider the recursive equations [14],[1] in the $q \rightarrow 1$ limit, simultaneously allowing $K \gg 1$ and keeping the number of endpoints q^K fixed. Such an idea has been suggested in [15], later worked out in more detail in [16]. Along that line an exact renormalization equation for continuous tree models has been derived in [17] for the general distribution of random variables.

The hierarchic tree models belong to the type of Ran-

dom Energy Models (REM's) which all share the leading term in the free energy with that of the simplest Derrida's model. The investigation of REM like models in finite dimensions has been started in [18] and got serious impetus from recent solution of 1-d models with logarithmic correlation of energies at different sites [19, 20]. That solution essentially used the generalization of Selberg integrals [21],[22] to complex number of integrations (see rigorous mathematical justification in [23]). The 1d models are intimately connected with the model [24],[25]. Recently [26] we have developed a statistical physics approach to related dynamical MSM models [27]. While all three types of REM models: hierarchical tree [1], 1-d logarithmic REM [20] and MSM [27] have exactly the same mean free energy as the standard REM, they have different distributions for the free energy and the partition function. Most essentially, in the Derrida's REM the probability distribution of the partition function has no fat tails in the high temperature phase [6], while such tails are present in 1d models [19, 20]. In this work we give indication that such tails exists in the hierarchic model, and also reveal the multi-fractal properties of the later, which is important for applications [7]-[10]. For a recent discussion of multifractal properties of REM, and other models with logarithmic correlations see [28, 29] and references therein.

In multifractal approach [30],[31] one considers the moments of a random variable (partition function) Z at some scale l defined with respect to the maximal scale L :

$$\langle Z^n \rangle = e^{\xi(n) \ln \frac{L}{l}} \hat{Z}_n, \quad (1)$$

where the exponent $\xi(n)$ defines the multi-scaling.

Knowing the $\xi(n)$ and the coefficients $\hat{Z}_n \sim O(1)$ in the limit of large L , one can reconstruct the probability distribution of the partition function. In this way such a distribution has been explicitly derived for 1-d logarithmic REMs [19, 20]

In the present article we will calculate $\xi(n)$ for hierar-

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chical tree, and derive recursive relations to define \hat{Z}_n , which, in principle, could be applied to calculate the probability distribution of Z within some accuracy. We shall see indications of the fact that the divergence of moments is essentially the same as for $1d$ case, thus the two types of models must share the same fat tails.

II. THE CALCULATION OF THE MOMENTS

A. The model with Normal distribution of random variables

Let us define the model outlined in the introduction following the papers [15],[17]. Starting with the hierarchical tree with integer branching number q and q^K end-points, we consider two such end-points w_i and w_j . The two paths connecting these points to the root coincide up to a level m , counting from the root. Accordingly, we can define the hierarchic distance between the two points as

$$v(w_i, w_j) = \frac{mV_0}{K} \quad (2)$$

where V_0 is defined as

$$q^K \equiv e^{V_0} \equiv L \quad (3)$$

Associated with every branch of the tree is a Gaussian random variable ϵ distributed according to the law

$$\rho\left(\frac{V_0}{K}, \epsilon\right) = \sqrt{\frac{K}{2V_0\pi}} \exp\left[-\frac{K\epsilon^2}{2V_0}\right] \quad (4)$$

We define the energy y_i at the endpoint w_i as a sum of corresponding variables ϵ sampled along the unique path connecting the point w_i to the of the tree.

We further define the partition function

$$Z = \sum_i e^{-\beta y_i - V_0 \frac{\beta^2}{2}}, \quad (5)$$

where the sum is over $e^V \equiv l$ end-points.

Then we obtain

$$\langle Z(e^V, b)^n \rangle = \sum_{i_1, \dots, i_n} \langle e^{-\beta y_{i_1} \dots - \beta y_{i_n}} \rangle \quad (6)$$

In Eq.(6) the sum is over all endpoints of the tree separated by the hierarchic distance V from a chosen point.

Now we consider the limit

$$q \rightarrow 1 \quad (7)$$

while keeping the value of V_0 fixed and large. Recall that $V_0 = K \log q$. In such a limit there are e^v end points at the hierarchic distance v . While calculating the $\langle Z^n \rangle$, we may discard the contributions given by Fig. 1b.

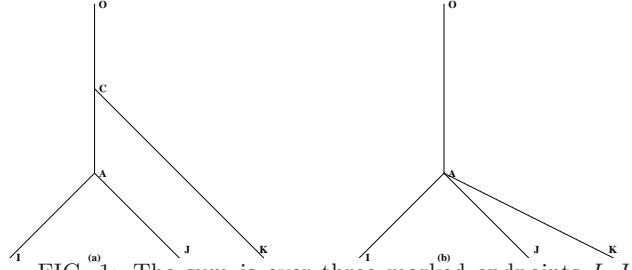


FIG. 1: The sum is over three marked endpoints I, J, K of the hierarchic tree. Only the Fig. 1.a contributes to the leading order. The locations of I, J, K given by Fig. 1.b yield negligible $O(q-1)$ contribution to the partition function in Eq.(8).

We formally replace the sum over the tree with the integration over a measure dw [15, 16],

$$\langle Z(e^V, b)^n \rangle = \prod_{i=1}^n \int dw_i \langle e^{-\beta \sum_i y(w_i)} \rangle \quad (8)$$

We are able to calculate $\langle Z^n \rangle$ for the integration range going over the maximal distance V . When calculating the correlations in Eq.(8) we use the following trick: the part of the trajectory of w_i with the hierarchic length v which has no overlap with trajectories from other points yields a factor $e^{\beta^2 v/2}$, while the part of the trajectory common to n such trajectories and hierarchic length v yields a factor $e^{n^2 \beta^2 v/2}$.

Consider $n = 2$ for simplicity, with summation going up to the maximal distance V . We then have e^{V-v} positions for the top level of hierarchy. Then, summing (or "integrating") over the number of two points at the distance e^{2v} we get

$$\begin{aligned} \langle Z(e^V, b)^2 \rangle &= \int_0^V dv e^{V-v} e^{2(V_0-v)\beta^2 + v\beta^2 - V_0\beta^2} \\ &= \frac{1}{1 - \beta^2} e^{V(2-\beta^2) + V_0\beta^2} = \\ &= \frac{1}{1 + b} e^{V(2+b) - bV_0} \end{aligned} \quad (9)$$

where we have introduced the parameter $b = -\beta^2$.

Consider now the integration in Eq.(8) going up to the maximal hierarchic distance V . We assume the following Ansatz:

$$\langle Z(e^V, b)^n \rangle \equiv e^{(V-V_0)(n+bn(n-1)/2)} e^{nV_0} \hat{Z}_n \quad (10)$$

where we have introduced dimensionless coefficients \hat{Z}_n . The results of our calculation support the ansatz in Eq.(10), see Eqs.(13)-(14).

Actually the ansatz Eq.(10) is correct only when

$$(n-1) - n \frac{b}{2} > -\frac{n^2 b}{2}, \quad (11)$$

otherwise there is a phase transition to a different phase [6], which manifests itself via diverging integration in Eq.(8).

We will explicitly consider the case of positive b (imaginary β) while calculating some eventual expression (there is no any restriction on n like Eq.(11)), like the analytical approximation of the fractional moment

$$< Z^\alpha(V, b) >, \quad (12)$$

and then continue analytically the resulting expressions to the realistic case of negative $b < 0$. The analytical continuation gives wrong results in SG, when we continue the expressions for moments to the other statistical physics phase. Hopefully in our case we are interested to continue to expressions of moments in the same phase.

Considering now the general case we calculate Z_n recursively. At the highest hierarchy level, n points are split into two groups with m and $(n-m)$ points, $1 \leq m < n$ accordingly, see Fig.2. Having the expressions of Z_m, Z_{n-m} at our disposal, we can calculate Z_n . We assume the minimum hierarchy level where all the paths from the root to n points meet is given by v . There are e^{V-v} such locations on the tree. Integrating over the positions of all points with a given v we arrive to

$$Z^n(e^V, b) = \sum_{1 \leq m < n} \times \int_0^V dv [Z_m(v) Z_{n-m}(v)] e^{V-v+bm(n-m)} \quad (13)$$

Using the scaling Ansatz Eq.(10) gives

$$\begin{aligned} \hat{Z}_n(b) &= \sum_{1 \leq m < n} \times \\ &= \frac{\hat{Z}_m(b) \hat{Z}_{n-m}}{n-1 + \frac{b}{2}(m(m-1) + (n-m)(n-m-1) - 2m(n-m))} \\ &= \frac{\sum_{1 \leq m < n} \hat{Z}_m(b) \hat{Z}_{n-m}(b)}{(n-1)(1+nb/2)} \end{aligned} \quad (14)$$

Thus, the problem amounts to solving the recursive equation (14) with the initial condition (12).

Eq.(10) defines the so-called multifractal scaling. We can rewrite it in the form

$$< Z(l, b)^n > \equiv e^{(n+bn(n-1)/2) \ln(l/L)} \hat{Z}_n \quad (15)$$

Let us now define a characteristic function

$$u(x) = \sum_{n=1} \hat{Z}_n x^n \quad (16)$$

for those values of β where the sum converges. We rewrite Eq.(14) as

$$(n-1)(1+nb/2) \hat{Z}_n(b) = \sum_{1 \leq m < n} \hat{Z}_m(b) \hat{Z}_{n-m}(b) \quad (17)$$

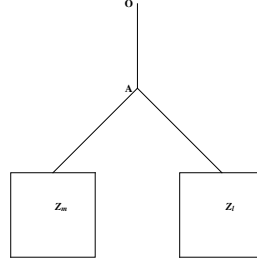


FIG. 2: The graphics for calculation of Z_n recursively fracturing $n = l + m$. The groups with l and m endpoints meet at the hierarchic level v at the point A.

Using the equations:

$$\begin{aligned} \sum_n x^n \sum_{1 \leq m < n} \hat{Z}_m(b) \hat{Z}_{n-m}(b) &= \left(\sum_n x^n \hat{Z}_n(b) \right)^2, \\ \sum_n n \hat{Z}_n(b) x^n &= x \frac{d}{dx} \sum_n \hat{Z}_n(b) x^n, \\ \sum_n n^l \hat{Z}_n(b) x^n &= \left(x \frac{d}{dx} \right)^l \sum_n \hat{Z}_n(b) x^n, \end{aligned} \quad (18)$$

we derive the following ODE for the characteristic function:

$$\begin{aligned} \frac{bx^2}{2} \frac{d^2 u(x)}{dx^2} + x \frac{du}{dx} &= u(x) + u^2(x), \\ u(0) &= 1, u'(0) = 1 \end{aligned} \quad (19)$$

Unfortunately we could not solve Eq.(19) to derive explicitly analytical expressions for Z_n in case of general (complex) values of n , which precludes from following the procedure of [19, 20]. Nevertheless, Eq.(19) allows us to extract a few first moments. We find:

$$\begin{aligned} \hat{Z}_1 &= 1, \hat{Z}_2 = \frac{1}{1+b}; \hat{Z}_3 = \frac{2}{(1+b)(2+3b)} \\ \hat{Z}_4 &= \frac{4}{(1+b)(2+3b)(3+6b)} + \frac{1}{(1+b)^2(3+6b)} \end{aligned} \quad (20)$$

The above relation allows us also to calculate the asymptotic of \hat{Z}_n at large n . In doing this we assume that our function $u(x)$ has a singularity at some $\bar{\rho}$ of the form [34]

$$u(x) = \frac{c}{(1 - \frac{x}{\bar{\rho}})^\alpha} \quad (21)$$

Putting the latter expression into Eq.(19), we derive:

$$\begin{aligned} \alpha &= 2, \\ c &= \frac{3b^2}{\bar{\rho}^2} \end{aligned} \quad (22)$$

Eq.(22) gives the asymptotic expression

$$\hat{Z}_n = \frac{3(n+1)b^2}{\bar{\rho}^{2+n}} \quad (23)$$

We can define the value of $\bar{\rho}$ only numerically, looking at Z_n for large values of the parameter n . A similar problem has been considered in [35]. Slightly modifying their results we arrive at the following algorithm to calculate $\bar{\rho}$. For a given integer n_0 we are looking the minimum over j and obtain:

$$\bar{\rho} = \min \left[\frac{6(n-1+n(n-1)b^2/2)}{n(n+2)p_j} \right]^{1/(2+n)} \big|_{0 \leq j \leq n_0} \quad (24)$$

It is possible to get accurate values of $\bar{\rho}$ by increasing the value of n_0 .

In [15],[17] the following reaction-diffusion equation has been derived for the same model:

$$\frac{\partial G(x, v)}{\partial v} = \frac{\partial^2 G(x, v)}{2\partial x^2} + G(x, v) \ln G(x, v) \quad (25)$$

where $0 \leq v \leq V$ played the role of time parameter in the reaction-diffusion equation. We should solve that equation with the initial value of G

$$G(x, 0) = \exp[-e^{-\beta x}] \quad (26)$$

Therefore, there must be a relation between ODE (19) and PDE (25): solving Eq.(25) one can find the solutions of Eq.(19). Eq.(25) allows one to be investigated via the traveling wave approach [17] and $\beta^2 = 2$ is its critical point. The critical point of Eq.(19) should therefore correspond to $b = -2$, and is associated with the anticipated freezing transition from the high temperature phase to spin-glass-like phase.

B. The case of general distribution

Instead of Eq.(4) we now consider

$$\rho\left(\frac{V_0}{K}, x\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \exp\left[\frac{V_0}{K}\phi(ih) - ihx\right] \quad (27)$$

Then for the the sum y_i of K random variables x , we can compose the distribution $\rho(V_0, \epsilon)$ simply by multiplying ϕ in the exponent of Eq. (27) by K :

$$\rho(V_0, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \exp[V_0\phi(ih) - ih\epsilon] \quad (28)$$

We define

$$Z = \sum_i e^{-\beta y_i - V_0\phi(\beta)}, \quad (29)$$

We are calculating the correlations in $\langle Z^n \rangle$ using the following trick: the part v of the trajectory of w_i , which belongs only him (no intersection with trajectories of other points), gives a factor $e^{\phi(\beta)v}$, while the part v of trajectory common to n trajectories, gives the factor $e^{\phi(n\beta)v}$.

Repeating the calculations of the previous section we obtain:

$$\langle Z^2 \rangle = \int_0^V dv e^{(V-v)\phi(2\beta)} e^{2v\phi(\beta) - 2V_0\phi(\beta)} e^{2v} e^{V-v} \quad (30)$$

The factor $e^{(V-v)\phi(2\beta)}$ corresponds to the part of trajectories $[A, O]$, see Fig1.a, while any of the lines $[I, A]$ and $[J, A]$ gives the factor $e^{2v\phi(\beta)}$, with e^{2v} being the number of possible locations of I, J at the hierarchic distance v , whereas e^{V-v} is the number of possible positions of the point A .

Performing the integration over all $0 \leq v \leq V$, we arrive at

$$Z_2(e^V, \beta) = \hat{Z}_2(\beta) e^{(V-V_0)(2\phi(\beta) - \phi(2\beta) + 1)} \quad (31)$$

$$\hat{Z}_2(\beta) = \frac{1}{1 - \phi(2\beta) + 2\phi(\beta)}$$

Let us derive recursive equations to calculate $Z_n, n > 1$, defined as

$$Z_n(\beta, V) \equiv e^{(V-V_0)(n\phi(\beta) - \phi(n\beta) + n)} e^{nV_0} \hat{Z}_n(\beta) \quad (32)$$

We need to consider all possible splitting $n = m + (n-m)$, with $1 \leq m \leq n$. Let us assume that two group with m and $n-m$ endpoints of our hierarchic tree are separated by the hierarchic distance v , and their trajectories meet at some point A . The calculations similar to those used to derive Eq.(14) give:

$$\hat{Z}_n(\beta, V) = \sum_{1 \leq m \leq n} \hat{Z}_m(\beta, v) \hat{Z}_{n-m}(\beta, v) \times e^{(V-v)(1 + \phi(n\beta) - n\phi(\beta))} \quad (33)$$

Then the scaling in Eq.(32) yields:

$$\hat{Z}_n(\beta) = \frac{\sum_{1 \leq m \leq n} \hat{Z}_m(\beta) \hat{Z}_{n-m}(\beta)}{n - 1 - \phi(n\beta) + n\phi(\beta)} \quad (34)$$

One can use Eqs.(31),(34) to calculate any positive integer moment of the partition function under the condition

$$n\phi(\beta) + n < 1 + \phi(n\beta) \quad (35)$$

Eq.(32) implies the multifractal behavior with

$$\xi(n) = n\phi(\beta) - \phi(n\beta) + 1 \quad (36)$$

For the corresponding generating function we now obtain an equation

$$(1 + \phi(\beta))x \frac{du(x)}{dx} - \phi(\beta x) \frac{d}{dx} u = u(x) + u^2(x), \quad u(0) = 1, u'(0) = 1 \quad (37)$$

For the same model in [17], another equation has been derived:

$$\frac{\partial G(x, v)}{\partial v} = \phi(\partial x) G(x, v) + G(x, v) \ln G(x, v) \quad (38)$$

Thus two equations must be related.

C. Comparison of Logarithmic 1-d REM and hierarchic model with normal distribution

In [20] the following partition function has been considered:

$$Z = \epsilon^{\hat{\beta}^2} \int_0^1 dy e^{\hat{\beta} y(x)},$$

$$\langle y(x)y(x') \rangle = 2 \ln \frac{|x - x'|}{\epsilon}, \quad (39)$$

with the parameter ϵ being an ultraviolet cutoff needed to regularize the model.

The above model shares the multifractal properties with our model for normal distribution of random energies, with the mapping

$$\epsilon = 1/e^{V_0},$$

$$2\hat{\beta}^2 = \beta^2 \quad (40)$$

Let us compare the moments. Ref.[20] gives the following expression for the moments:

$$\hat{Z}_n = \prod_{j=1}^{j=n} \frac{\Gamma(1 - (j-1)\gamma)^2 \Gamma(1 - \gamma j)}{\Gamma(2 - (n+j-2)\gamma) \Gamma(1 - \gamma)} \quad (41)$$

where $\gamma = \hat{\beta}^2$.

We have, naturally $\hat{Z}_1 = 1$ and further

$$\hat{Z}_2 = \frac{\Gamma(1 - 2\gamma)}{\Gamma(2 - \gamma)\Gamma(2 - 2\gamma)} = \frac{1}{(1 - \gamma)(1 - 2\gamma)}$$

$$\hat{Z}_3 = \frac{\Gamma(1 - 2\gamma)^3 \Gamma(1 - 3\gamma)}{\Gamma(2 - 2\gamma)\Gamma(2 - 3\gamma)\Gamma(2 - 2\gamma)}$$

$$\hat{Z}_4 = \frac{\Gamma(1 - 2\gamma)^3 \Gamma(1 - 3\gamma)^3 \Gamma(1 - 4\gamma)}{\Gamma(1 - \gamma)\Gamma(2 - 3\gamma)\Gamma(2 - 4\gamma)\Gamma(2 - 5\gamma)} \quad (42)$$

Comparing with the results of the section II-A, we see that second moment \hat{Z}_2 has different expansion for the small β^2 ,

$$\hat{Z}_2 \approx 1 + 3\gamma, \quad (43)$$

More important information contained in moments is however the *smallest* real pole γ_n as a function of γ (respectively, b). Indeed, precisely those poles define the critical temperatures $T_n = \gamma_n^{-1/2}$ below which the given moment of the partition function start to diverge. It is easy to see from Eq. (41) that for one-dimensional model $\gamma = 1/n$, $n = 2, 3, \dots$ while Eq.(42) gives the smallest pole poles at $b/2 = 1/2$ for Z_2 , $b/2 = 1/3$ for Z_3 , $b/2 = 1/4$ for Z_4 , etc. The results for a few lower moments indicate that the two models share the same sequence of "transition temperatures". Though we are unable to prove this statement in full generality, the factor $(1 - bn/2)$ appearing in the denominator of the right-hand side of the recursion Eq.(14) makes the statement very plausible, if no special cancelations happen. Assuming that correspondence is correct, the latter property would imply the probability density for the

partition function in both models share the same power law tail: $\mathcal{P}(Z) \propto Z^{-1-\frac{1}{\beta^2}}$ valid everywhere in the high-temperature phase $\beta < \beta_c = 1$. Such a tail was indeed proposed as one of the universal features characterizing the class of models with logarithmic correlations [20].

We obtained analytical expressions for the hierarchic model's Z_n at positive b , Eq.(14). Having large series of Z_n for positive b , one can try to construct an approximate expression for the probability distribution and fractional moments.

III. CONCLUSION

The investigation of the statistical mechanics of REM-like models in finite dimension and their multifractal properties remains an active field of research, especially after the advances achieved in [27] and [20]. In this article we confirmed that the multifractal properties are shared by the third class of REM-like models: directed polymers on a hierarchic tree. We calculated the corresponding moments important for the applications. In particular, we use those moments to conjecture the power law tail of the distribution of partition function Z in the high-temperature phase. Unfortunately it is impossible to derive exact probability distribution, as has been done in [20] for the 1-d logarithmic REM. Nevertheless, our formulas allow the calculation of infinite series of the moments for some values of parameters. We hope that such information could be applied to recover the probability distribution, a well known problem in probability theory [32].

Our results are interesting for the mathematics of reaction-diffusion equations as well, as we related them to a certain nonlinear ODE. Moreover, we found the connection between the multifractal spectrum and the hierarchic tree model Eq.(36), while the latter has been mapped to some reaction-diffusion equations [17], see Eq.(38) in the present article. Thus the reaction -diffusion equations have some multiscaling structures. One can connect reaction diffusion equation and nonlinear ODE with the interesting versions of multi-fractal phenomenon, i.e. [37], and try to investigate them for deriving the multifractal scaling from the reaction diffusion equation (38). Such a work is currently in the progress.

The directed walk models on hierarchic tree describes a rather rich physics, as well as some connections with quantum models in finite dimension. The relation to 2-d conformal field models is rather well known [12],[13], and the work [17] mentions the relations to quantum disorder problem in finite dimensions [36]. It will be interesting to try to apply the methods of the current work or our renormalization group (38) to the finite dimensional quantum disorder problem.

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